7. Tēma: Polinomi ar veseliem koeficientiem

Uzdevums 7.1 (IMO1982.4): Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers x, y, then it has at least three such solutions. Show that the equation has no solutions in integers for $n = 2891$.

> **lv:** Dots tāds naturāls skaitlis n, ka vienādojumam
>
> $$x^3 - 3xy^2 + y^3 = n$$
>
> ir atrisinājums veselos skaitļos x, y. Pierādīt, ka tad šim vienādojumam ir vismaz trīs dažādi atrisinājumi. Pierādīt, ka vienādojumam nav atrisinājumu, ja $n = 2891$.

Uzdevums 7.2 (IMO1987.3): Let $n \geq 2$ be an integer. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq \sqrt{\frac{n}{3}}$, then $k^2 + k + n$ is prime for all integers k such that $0 \leq k \leq n - 2$.

> **:** Dots naturāls skaitlis $n \geq 2$ un zināms, ka $k^2 + k + n$ ir pirmskaitlis visiem k, kuriem $0 \leq k \leq \sqrt{\frac{n}{3}}$. Pierādīt, ka tad $k^2 + k + n$ ir pirmskaitlis visiem k, kuriem $0 \leq k \leq n - 2$.

Uzdevums 7.3 (IMO1994.4): Find all ordered pairs (m, n) where m and n are positive integers such that $\frac{n^3+1}{mn-1}$ is an integer.

> **lv:** Atrast visus sakārtotus naturālu skaitļu pārus (m, n), kur $\frac{n^3+1}{mn-1}$ ir vesels skaitlis.

Uzdevums 7.4 (IMO1998.4): Determine all pairs (x, y) of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$.

> **lv:** Atrast visus naturālu skaitļu pārus (x, y), kuriem $x^2y + x + y$ dalās ar $xy^2 + y + 7$.

Uzdevums 7.5 (IMO2003.2): Determine all pairs of positive integers (m, n) such that

$$\frac{m^2}{2mn^2 - n^3 + 1}$$

is a positive integer.

> **lv:** Atrast visus naturālu skaitļu pārus (m, n) tādus, ka $\dfrac{m^2}{2mn^2 - n^3 + 1}$ ir naturāls skaitlis.

Uzdevums 7.6 (IMOSHL.2001.N5): Let $a > b > c > d$ be positive integers and suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

Uzdevums 7.7 (IMOSHL.2002.N6): Find all pairs of positive integers $m, n \geq 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself an integer.

Uzdevums 7.8 (IMOSHL.2003.N3): Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

Uzdevums 7.9 (IMOSHL.2005.N7): Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, where $a_0, \ldots, a_n$ are integers, $a_n > 0$, $n \geq 2$. Prove that there exists a positive integer m such that $P(m!)$ is a composite number.

Uzdevums 7.10 (IMOSHL.2006.N4): Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients and let k be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where P occurs k times. Prove that there are at most n integers t such that $Q(t) = t$.

Uzdevums 7.11 (IMOSHL.2009.N2): A positive integer N is called balanced, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b, consider the polynomial P defined by $P(x) = (x + a)(x + b)$.

(a) Prove that there exist distinct positive integers a and b such that all the number $P(1)$, $P(2)$,. . ., $P(50)$ are balanced.

(b) Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$.

Uzdevums 7.12 (IMOSHL.2009.N5): Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \geq 1$, where T^n denotes the n-fold application of T.

Uzdevums 7.13 (IMOSHL.2010.N3): Find the smallest number n such that there exist polynomials $f_1, f_2, \ldots, f_n$ with rational coefficients satisfying

$$x^2 + 7 = f_1(x)^2 + f_2(x)^2 + \ldots + f_n(x)^2.$$

Uzdevums 7.14 (IMOSHL.2010.N4): Let a, b be integers, and let $P(x) = ax^3 + bx$. For any positive integer n we say that the pair (a, b) is n-good if $n|P(m) - P(k)$ implies $n|m - k$ for all integers m, k. We say that (a, b) is *very good* if (a, b) is n-good for infinitely many positive integers n.

(a) Find a pair (a, b) which is 51-good, but not very good.

(b) Show that all 2010-good pairs are very good.

Uzdevums 7.15 (IMOSHL.2011.N2): Consider a polynomial $P(x) = \prod_{j=1}^{9}(x+d_j)$, where $d_1, d_2, \ldots d_9$ are nine distinct integers. Prove that there exists an integer N, such that for all integers $x \geq N$ the number $P(x)$ is divisible by a prime number greater than 20.

Uzdevums 7.16 (IMOSHL.2011.N6): Let $P(x)$ and $Q(x)$ be two polynomials with integer coefficients, such that no nonconstant polynomial with rational coefficients divides both $P(x)$ and $Q(x)$. Suppose that for every positive integer n the integers $P(n)$ and $Q(n)$ are positive, and $2^{Q(n)} - 1$ divides $3^{P(n)} - 1$. Prove that $Q(x)$ is a constant polynomial.

Uzdevums 7.17 (IMOSHL.2012.N5): For a nonnegative integer n define $\operatorname{rad}(n) = 1$ if $n = 0$ or $n = 1$, and $\operatorname{rad}(n) = p_1 p_2 \cdots p_k$ where $p_1 < p_2 < \cdots < p_k$ are all prime factors of n. Find all polynomials $f(x)$ with nonnegative integer coefficients such that $\operatorname{rad}(f(n))$ divides $\operatorname{rad}(f(n^{\operatorname{rad}(n)}))$ for every nonnegative integer n.

Uzdevums 7.18 (LVTST1977.10.2): $P(x)$ ir polinoms ar veseliem koeficientiem; a, b un c - dažādi veseli skaitļi. Dots, ka $P(a) = P(b) = P(c) = 1$. Pierādīt, ka vienādojumam $P(x) = 0$ nav atrisinājumu veselos skaitļos.

Uzdevums 7.19 (LVTST1982.9.1): Atrisināt naturālos skaitļos vienādojumu sistēmu

$$\begin{cases} x^3 - y^3 - z^3 = 3xyz \\ x^2 = 2(y+z) \end{cases}$$

Uzdevums 7.20 (LVTST1994.9-12.1): Dots, ka x un y ir naturāli skaitļi un

$$3x^2 + x = 4y^2 + y.$$

Pierādīt, ka $x - y$, $3x + 3y + 1$ un $4x + 4y + 1$ ir naturālu skaitļu kvadrāti.

Uzdevums 7.21 (LVTST2009.9-12.1): Kādiem naturāliem skaitļiem m un n, kas abi lielāki par 1, skaitlis $n^3 - 1$ dalās ar $m \cdot n - 1$?

Uzdevums 7.22 (BwTst2015.Day1.3): Pierādīt, ka neeksistē polinoms $P(x)$ ar veseliem koeficientiem un naturāls skaitlis m, tādi, ka

$$x^m + x + 2 = P(P(x))$$

izpildās visiem veseliem skaitļiem x.

Uzdevums 7.23 (Bw2015.18): Dots n-tās pakāpes polinoms $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ ar pakāpi $n \geq 1$, kuram ir n (ne obligāti dažādas) veselas saknes. Pieņemam, ka eksistē tādi atšķirīgi pirmskaitļi $p_0, p_1, \ldots, p_{n-1}$, ka $a_i > 1$ ir p_i pilna pakāpe visiem $i = 0, \ldots, n-1$. Atrast visas iespējamās n vērtības.

Uzdevums 7.24 (BwTst2016.Day2.8): Atrisināt veselos skaitļos vienādojumu sistēmu:

$$\begin{cases} a^3 & = & abc + 2a + 2c \\ b^3 & = & abc - c \\ c^3 & = & abc - a + b \end{cases}$$

Uzdevums 7.25 (Bw2016.4): Dots naturāls skaitlis n un tādi veseli skaitļi a, b, c, d, ka gan $a + b + c + d$, gan $a^2 + b^2 + c^2 + d^2$ dalās ar n. Pierādīt, ka arī $a^4 + b^4 + c^4 + d^4 + 4abcd$ dalās ar n.